Vacuum field energy densities and Casimir-Polder forces near a fluctuating boundary

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The Casimir effect is an attractive force of electromagnetic nature between two neutral metallic or dielectric plates in the vacuum.

It is a consequence of the change of the zero-point energy of a field due to the presence of boundary conditions.

The Casimir force is an observable manifestation of the quantum zero-point fluctuations of the radiation field, even at the macroscopic level.

Usually boundary conditions are supposed fixed in space (static Casimir force) or subject to a prescribed movement by an external action (dynamical Casimir effect).

Casimir and Casimir-Polder forces are also related to the vacuum energy density of the electromagnetic field.
Vacuum electromagnetic energy density near a fixed boundary (perfectly conducting plate), related to the atom-wall Casimir-Polder force

\[
\langle E^2(z) \rangle_R = \frac{3c\hbar}{4\pi z^4}
\]

\[
\langle B^2(z) \rangle_R = -\frac{3c\hbar}{4\pi z^4}
\]

Renormalized electric and magnetic energy densities and renormalized field fluctuations diverge at the boundary \( z=0 \).

*Even a more realistic model of the conducting plate, such as the plasma model, does not eliminate the surface divergence.*
**Dielectric to conductor limit**: extra singular terms at the boundary \((z=0)\)

Nondispersive and nondissipative dielectric half-space with refractive index \(n\)

\(\rightarrow\) Quantization in terms of the Carniglia-Mandel triplets

**Ideal conductor limit** \(n \rightarrow \infty\)

\[
\left\langle E^2(z) \right\rangle_{\eta R}^{\text{con}} = \lim_{n \rightarrow \infty} \left\langle E^2(z) \right\rangle_{\eta} - \lim_{n \rightarrow 1} \left\langle E^2(z) \right\rangle_{\eta} = \frac{4 \epsilon_0 \hbar}{\pi} \frac{12 z^2 - c^2 \eta^2}{\left(4 z^2 + c^2 \eta^2\right)^3}
\]

\[
\left\langle B^2(z) \right\rangle_{\eta R}^{\text{con}} = \lim_{n \rightarrow \infty} \left\langle B^2(z) \right\rangle_{\eta} - \lim_{n \rightarrow 1} \left\langle B^2(z) \right\rangle_{\eta} = -\frac{4 \epsilon_0 \hbar}{\pi} \frac{12 z^2 - c^2 \eta^2}{\left(4 z^2 + c^2 \eta^2\right)^3}
\]

\(\eta\) is an ultraviolet cut-off frequency introduced by a time-splitting procedure
\[ \langle E^2 \rangle^\text{con}_{\eta R} \]

\[ \frac{1}{\eta} = 2 \cdot 10^{16} \text{ s}^{-1} \]
The dielectric to ideal conductor limiting procedure shows a complicated structure of vacuum field fluctuations and energy densities near the boundary.

$\eta \to 0$: the well-known $z^{-4}$ behaviour is recovered for $z\neq 0$, but extra singular terms are present at the boundary ($z=0$).

The surface energy density singularities contain a finite amount of energy.

They contribute to the electromagnetic self-energy of the system.

A rough surface can make smooth the surface divergences.

Surface divergences can be also softened by fluctuating boundaries.

If the mechanical degrees of freedom of the wall are treated quantum mechanically, the wall has position fluctuations which affect field fluctuations in proximity of the boundary.

Casimir and Casimir-Polder interactions are significantly affected by the wall’s motion.
What happens to Casimir energies and forces if the cavity walls can move, and their movement is treated quantum mechanically?

→ Quantum fluctuations of the position of the mobile walls and of the cavity length.

**1D scalar case**

Massless scalar field in a 1D cavity with one mobile wall of mass $M$ (zero temperature). The wall is subjected to radiation pressure. The mobile wall is also subjected to a harmonic potential with frequency $\omega_{osc}$. $L_0$ is its equilibrium position.
Law Hamiltonian (valid for small displacements of the wall from its equilibrium position $L_0$, and expressed in terms of field operators relative to the wall’s equilibrium position)

\[ H = H_0 + H_I \]

\[ H_0 = \hbar \omega_{osc} b^\dagger b + \sum_k \hbar \omega_k a_k^\dagger a_k \]

mirror Hamiltonian

field Hamiltonian

\[ H_{int} = -\sum_{kj} C_{kj} (b + b^\dagger)N \left[ (a_k + a_k^\dagger)(a_j + a_j^\dagger) \right] \]

\[ C_{kj} = (-1)^{k+j} \left( \frac{\hbar}{2} \right)^{3/2} \frac{1}{L_0 \sqrt{\mathcal{M}}} \sqrt{\frac{\omega_k \omega_j}{\omega_{osc}}} \]
coupling constant between the field and mirror’s mechanical degrees of freedom

Mirror’s motion introduces a field-mirror interaction and an effective interaction between the field modes.
Ground state of the mirror-field interacting system

\[ |g\rangle = |\{0_p\}, 0\rangle + \sum_{kj} D_{kj} |\{1_k, 1_j\}, 1\rangle \]

pairs of field excitations  mirror’s excitation

Virtual field excitations inside the cavity, due to the quantum fluctuations of the position of the mobile wall.

Average photon number and wall’s excitation number

\[ \langle g | a_m^{\dagger} a_m | g \rangle = \sum_j \frac{\hbar}{2L_0^2 M} \frac{\omega_m \omega_j}{\omega_{osc}} \frac{1}{(\omega_{osc} + \omega_m + \omega_j)^2} \] \[ \langle g | b^{\dagger} b | g \rangle = \sum_{jk} \frac{\hbar}{4L_0^2 M} \frac{\omega_k \omega_j}{\omega_{osc}} \frac{1}{(\omega_{osc} + \omega_k + \omega_j)^2} \]

These expressions need to be regularized by an ultraviolet cut-off

The contribution of a single pair of virtual photons is maximum when the sum of their frequencies is equal to \(\omega_{osc}\) (compare with the dynamical Casimir effect).
Photon spectrum \( (\omega_{\text{osc}} = 10^5 \text{ s}^{-1}, L_0 = 10 \mu\text{m}) \)

The average photon number in the cavity is very small but, for reasonable values of the parameters involved, comparable to that in other observable quantum electrodynamical effects such as the Lamb shift.
Energy shift due to the mirror-field interaction (that adds to the usual Casimir energy for fixed walls):

\[ E_g^{(2)} = - \sum_{kj} \frac{\hbar^2}{4L_0^2M} \frac{\omega_k \omega_j}{\omega_{osc}} \frac{1}{(\omega_{osc} + \omega_k + \omega_j)} \]

This energy gives an extra contribution to the Casimir force between the two walls, that originates from the (quantum) position fluctuations of the mobile wall.

Casimir force for fixed walls:

\[ F_c = \frac{\pi \hbar c}{24L_0^2} \]
The correction of the Casimir force related to the interaction between field and mechanical degrees of freedom, is proportional to $1/M$.

Numerical evaluation yields a change of the Casimir force of a few percent when:

$$\omega_{osc} = 10^5 \text{ s}^{-1}, L_0 = 1 \text{ \mu m}, M = 10^{-21} \text{ kg}, \omega_{cut} = 10^{16} \text{ s}^{-1}$$

Very small values of the mirror mass can be obtained in modern quantum optomechanics experiments, and this suggests possibility of observing this tiny effect.
**Field energy density inside the cavity with the mobile wall:**

Field energy density operator: 
\[
H(x) = \frac{1}{2} \left[ \frac{1}{c^2} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \left( \frac{d\phi}{dx} \right)^2 \right]
\]

**Renormalized ground-state energy density for a fixed-wall cavity**

\[
\langle H^R \rangle_0 = -\frac{\pi c\hbar}{24L_0^2}
\]

**Renormalized ground-state energy density for the cavity with the mobile wall:**

\[
\langle g \mid H \mid g \rangle = \langle H^R \rangle_0 + \Delta H(x)
\]

\[
\Delta H(x) = \frac{\hbar^2}{2L_0^3 M \omega_{osc}} \sum_j \sum_{kp} (-1)^{k+p} \frac{\omega_k \omega_j \omega_p}{(\omega_{osc} + \omega_k + \omega_j)(\omega_{osc} + \omega_p + \omega_j)} \cos\left[\frac{\pi(k - p)}{L_0} x\right]
\]
Change of the energy density due to the mirror motion, in the very proximity of the mobile wall (sharp cut-off)

\[ M = 10^{-11} \text{ kg}, \quad \omega_{osc} = 10^5 \text{ s}^{-1}, \quad L_0 = 10 \text{ \mu m} \]

The effect of the motion of the wall on the field energy density is relevant in the proximity of the mobile wall, but it is negligible far from it (virtual photons are localized near the wall).

It depends on the cut-off frequency and on the specific model for a real surface.

The energy density diverges at the interface in the ideal conductor limit

\[ \omega_{\text{cut}} \to \infty \]

This may appear in contradiction with the fact that the wall has position fluctuations, that should smear out surface divergences.

Field quantization has been done in terms of operators relative to the mirror’s equilibrium position \( L_0 \).

\[ \rightarrow \text{Average of the field energy density over the probability distribution of the mirror’s position.} \]
Reduced density operator for the wall

\[ \rho_{osc} = (1 - p_1)|0\rangle\langle 0| + p_1|1\rangle\langle 1| \]

Distribution probability for the position \( q \) of the mobile wall

\[ f(q) = (1 - p_1)f_0(q) + p_1f_1(q) \]

\[ p_1 = \frac{\hbar}{4L_0^2M} \sum_{kj} \frac{\omega_k \omega_j}{\omega_{osc} \left( \omega_{osc} + \omega_k + \omega_j \right)^2} \]

\[ f_0(q) = \left( \frac{M \omega_{osc}}{\pi \hbar} \right)^{1/2} \exp\left( -\frac{M \omega_{osc}}{\hbar} q^2 \right) \]

\[ f_1(q) = \left( \frac{4}{\pi} \left( \frac{M \omega_{osc}}{\pi \hbar} \right)^3 \right)^{1/2} q^2 \exp\left( -\frac{M \omega_{osc}}{\hbar} q^2 \right) \]
Average value of the correction to the field energy density over
the distribution function of the mirror position (small displacements)

\[
\langle \langle \Delta H \rangle \rangle = \frac{\hbar^2}{2L_0^3M \omega_{osc}} \sum_{kjp} (-1)^{k+p} \frac{\omega_k \omega_j \omega_p}{(\omega_{osc} + \omega_k + \omega_j)(\omega_{osc} + \omega_p + \omega_j)} \sum_{n=0}^{1} p_n < \cos[(k_j - k_p)x] >_n
\]
Energy density near the mobile wall, averaged over its position (100 modes)

Blue curve: not averaged.
Violet: averaged over the mirror’s ground state.
Green: averaged over the mirror’s first excited state.
Blue: not averaged;
Violet: averaged over the mirror’s ground state;
Green: averaged over the mirror’s first excited state.

The average over the position of the mobile wall eliminates the surface divergence in the perfect conductor limit.
The energy density maximum is not necessarily located at the mirror’s equilibrium position.

The effect of the position fluctuations of the wall on the field energy density is particularly relevant in the vicinity of the mobile wall.

Virtual photons “emitted” by the wall are confined in its vicinity.

The energy-density change near the mobile wall rapidly grows with the cut-off frequency $\omega_{\text{cut}}$. It diverges at the wall’s surface when $\omega_{\text{cut}}$ goes to infinity.

The energy density inside the cavity can be experimentally probed through the Casimir-Polder type interaction energy with a polarizable test body:

$$\Delta E = -\frac{1}{2} \alpha \left\langle \dot{\phi}^2 (R) \right\rangle$$
Average value in the interacting ground state $|g\rangle$

\[
\left\langle \dot{\phi}^2 (x) \right\rangle^R_g = \left\langle \dot{\phi}^2 (x) \right\rangle^R_0 + \Delta \dot{\phi}^2 (x)
\]

\[
\left\langle \dot{\phi}^2 (x) \right\rangle^R_0 = \frac{\hbar c^3 \pi}{24 L_0^2} \left[ 3 \sin^{-2} \left( \frac{\pi}{L_0} x \right) - 1 \right] 
\]

(same as in the case of fixed wall)

correction due to the mirror-field interaction:

\[
\Delta \dot{\phi}^2 (x) = \sum_k \sum_{jp} \frac{\hbar^2 c^2}{2 L_0^3 M \omega_0} \left( -1 \right)^{j+p} \frac{\omega_k \omega_j \omega_p}{(\omega_{osc} + \omega_j + \omega_k)(\omega_{osc} + \omega_p + \omega_k)} \sin(k_j x) \sin(k_p x)
\]

Also in this case the correction is relevant in the proximity of the mobile wall.
Electromagnetic 1D case

Renormalized scalar Green’s function with Dirichlet boundary conditions

\[ G_R(x,t; x', t') = \langle g \phi_{BC}(x,t) \phi_{BC}(x', t') | g \rangle - \langle 0 | \phi_{un}(x,t) \phi_{un}(x', t') | 0 \rangle \]

\[ G_R(x,t; x', t') = G_{R0}(x,t; x', t') + \Delta G_R(x,t; x', t') \]

zeroth order in the atom-mirror coupling

first order in the atom-mirror coupling

\[ \Delta G_R(x,t; x', t') = \frac{8\hbar c^2}{L_0} \sum_{j \ell r} \frac{1}{(\omega_j \omega_r)^{1/2}} D_{j \ell} D_{\ell r} \cos(\omega_j t - \omega_r t') \sin(k_j x) \sin(k_r x') \]
The electric and magnetic energy densities can be obtained by applying appropriate differential operators to the scalar propagator

\[
\langle E_z^2(x) \rangle_0 = \lim_{(x,t') \rightarrow (x,t)} c^{-2} \partial_t \partial_{t'} \langle G_{R0} (x,t;x',t') \rangle
\]

\[
\langle B_y^2(x) \rangle_0 = \lim_{(x,t') \rightarrow (x,t)} c^{-2} \partial_x \partial_{x'} \langle G_{R0} (x,t;x',t') \rangle
\]

**Correction due to the effective wall-field interaction**

\[
\langle E_z^2(x) \rangle_1 = \lim_{(x',t') \rightarrow (x,t)} c^{-2} \partial_t \partial_{t'} \langle \Delta G_R (x,t;x',t') \rangle = \frac{\hbar^2}{L_0^3 M \omega_{osc}} \sum_{j \ell r} (-1)^{\ell+r} \times \frac{\omega_j \omega_\ell \omega_r}{(\omega_{osc} + \omega_j + \omega_\ell)(\omega_{osc} + \omega_j + \omega_r)} \sin(k_\ell x) \sin(k_r x)
\]

\[
\langle B_y^2(x) \rangle_1 = \lim_{(x',t') \rightarrow (x,t)} c^{-2} \partial_x \partial_{x'} \langle G_{R0} (x,t;x',t') \rangle = \frac{\hbar^2}{L_0^3 M \omega_{osc}} \sum_{j \ell r} (-1)^{\ell+r} \times \frac{\omega_j \omega_\ell \omega_r}{(\omega_{osc} + \omega_j + \omega_\ell)(\omega_{osc} + \omega_j + \omega_r)} \cos(k_\ell x) \cos(k_r x)
\]
Change of the electric, magnetic and total e.m. energy density due to the fluctuating motion of the conducting wall.

Change of the e.m. energy density near the mobile wall for different values of the cut-off frequency.

\[
\omega_{\text{cut}} = 6 \cdot 10^{15} \text{ s}^{-1} \text{ (blue line)}
\]

\[
\omega_{\text{cut}} = 8 \cdot 10^{15} \text{ s}^{-1} \text{ (red line)}
\]

\[
\omega_{\text{cut}} = 9 \cdot 10^{15} \text{ s}^{-1} \text{ (green line)}
\]

\[
\omega_{\text{cut}} = 10^{16} \text{ s}^{-1} \text{ (black line)}
\]
Scalar 3D case (Dirichlet boundary conditions)

Generalization of the effective Law Hamiltonian to 3D -> Effective Hamiltonian -> Corrected ground state -> Field energy density inside the cavity

Correction to the field energy density near the mobile wall

\[ L_0 = 10 \mu m; \quad L_y = L_z = 0.5 \times 10^{-4} \text{ m}; \quad M = 10^{-11} \text{ kg}; \]

\[ \omega_{osc} = 10^5 \text{ s}^{-1}; \quad \omega_{cut} = 10^{15} \text{ s}^{-1} \]

Conclusions and perspectives

• Field energy density divergences at a metal-vacuum interface: their role in the electromagnetic self-energy
• A massless scalar field in a 1D cavity with a mobile wall; Mirror’s mechanical degrees of freedom are treated quantum mechanically;
• In the ground state of the interacting system, pairs of virtual photons are generated inside the cavity
• Change of the Casimir energy and force between the two mirrors
• Change of the field energy density inside the cavity, particularly relevant in the proximity of the mobile wall
• Change of the Casimir-Polder interaction energy for a polarizable body placed inside the cavity.
• Extension to the 1D electromagnetic field
• Extension to the 3D scalar field

• Extension to the 3D electromagnetic field (TE and TM modes)
• Case of a real wall (plasma model)
• Precise estimate of the change of Casimir and Casimir-Polder interactions for a real metal or dielectric mobile mirror